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LIMIT CYCLES AND CHAOS IN EQUATIONS OF THE PENDULUM TYPE*

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It is proved that for sufficiently small ε the equation

$$x^{\cdot\cdot} + \sin x = \varepsilon x^{\cdot} \cos nx, \quad n \in N \tag{0.1}$$

where ε is a parameter, has exactly $n-1$ coarse limit cycles (l.c.'s) in the region of oscillatory motions and no l.c.'s in the region of rotary motions (i.e., l.c.'s going round the phase cylinder). This result is used to study an equation of type (0.1) with time-periodic term on the right. The role of l.c.'s in the formation of quasi-attractors (q.a.'s) is demonstrated. A computer-generated description is given of the process by which q.a.'s with developed chaos are formed (for $n=3$).

1. Statement of the problem. Main results. We consider equations of the form

$$x^{\cdot\cdot} + A(x) = \varepsilon f(x, x^{\cdot}, \nu t; \varepsilon) \tag{1.1}$$

where A is a 2π-periodic function of x and f a periodic function of x and $\varphi = \nu t$ with the same period; ε, ν are parameters. Equations of this kind govern the motions of various pendulums. Among other applications we mention the problem of the structure of resonance zones in non-conservative time-periodic systems

$$\frac{du}{d\tau} = \frac{\partial H(u,v)}{\partial v} + \mu R(u,v,\tau), \quad \frac{dv}{d\tau} = -\frac{\partial H(u,v)}{\partial u} + \mu G(u,v,\tau) \tag{1.2}$$

where μ is a small parameter. As shown in /1, 2/, this problem involves investigating an equation of the form (1.1) with a small parameter ε depending on μ. In addition, $f = \sigma(x) x^{\cdot} + O(\varepsilon)$, where $\sigma(x)$ is defined by the divergence of the vector field of system (1.2).

We set $A(x) = \sin x$ and consider, first of all, the case in which ε is a small parameter Eq.(1.1) has been studied in this case /3/ for a special form of the function f. A more general setting was considered in /2/. It has been observed that an important role in the study of Eq.(1.1) is played by the l.c.'s of the autonomous equation

$$x^{\cdot\cdot} + \sin x = \varepsilon f_0(x, x^{\cdot}), \quad f_0 = \langle f(x, x^{\cdot}, \varphi; 0)\rangle_{\varphi} \tag{1.3}$$

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Equations of type (1.3) crop up in a variety of applied problems. Among these are, for example, the problem of oscillations of rectangular bodies suspended from cables under the action of air flow /4/, selfoscillations of a synchronous motor /5/, and phase synchronization problems /6/.

The investigation of Eq.(1.3) reduces to the problem of l.c.'s. However, it is no easier to solve the problem of l.c.'s for Eq.(1.3) than to analyse the original Eq.(1.1). As yet the l.c. problem has not been solved (and neither has the problem of steady-state solutions and their stability) for Eq.(0.1), which may be considered, together with the Van der Pol equation, as one of the most fundamental equations of non-linear oscillation theory. In this paper we remedy this omission. We shall present a full qualitative investigation of Eq.(0.1), proving the following

Proposition. There exists a number $\varepsilon_*(n) > 0$ so small that for any $|\varepsilon| \in (0, \varepsilon_*)$ Eq.(0.1) with $n \geqslant 1$ has exactly $n-1$ coarse l.c.'s (of the first kind) in the region of oscillatory motions. In the region of rotary motions there are no l.c.'s (of the second kind).

Remark 1. The question of the stability of l.c.'s is fairly easy to settle. Indeed, stable and unstable l.c.'s alternate. Since the equilibrium state $x = x^{\cdot} = 0$ is an unstable focus for $0 < \varepsilon < 2$, the nearest l.c. surrounding the equilibrium state is stable, the next is unstable, and so on.

Remark 2. An equation of type (0.1) was considered in the oscillatory region in /4/. There the unperturbed solution was taken to be a harmonic mode, which is legitimate only for small x. The theoretical result of /4/ is thus valid not for Eq.(0.1) but for a quasilinear equation (with $\sin x$ on the left replaced by x).

Remark 3. A bound has been established /7/ for the number of l.c.'s of Eq.(1.3) in the case where f_0 is a trigonometric polynomial in x and an algebraic polynomial in $x^{\cdot}$ (in this connection see also /1, 8, 9/).

After proving the proposition we shall construct the equation

$$x^{\cdot\cdot} + \sin x = \varepsilon x^{\cdot} (\cos nx + a) \tag{1.4}$$

which has exactly n l.c.'s at $a = (-1)^n/(4n^2 - 1)$. Of these, $n-1$ lie in the region of oscillatory motions and one on the boundary between the oscillatory and rotary regions (saddle-type l.c.). If n is odd, the saddle-type l.c. is stable if $\varepsilon > 0$. Using this fact one can investigate the "chaotic dynamics" of the equation

$$x^{\cdot\cdot} + \sin x = \varepsilon x^{\cdot} (\cos nx + a)(1 + c \sin \nu t) \tag{1.5}$$

where ε, a, c, ν are parameters. The purpose of this study is to demonstrate the role of l.c.'s in the formation of non-trivial attracting sets. Hyperbolic (non-trivial) attracting sets are usually known as strange attractors. For this class of equations, attracting sets may possibly contain stable points with a large period. In this connection we shall use the term "quasi-attractor", which is not unknown in the literature, and sometimes also the more common term "chaos".

The solution of the problem involved the use of computers. Some of the results are shown in Figs.1-6. Figs.1 and 2 illustrate the trajectories of Eq.(1.4), and Figs.3-5 the Poincaré map for Eq.(1.5) in the region $|x| \leqslant \pi$, $|x^{\cdot}| \leqslant 3$, which is part of the development of the phase cylinder $\{x \bmod 2\pi, x^{\cdot}\}$. An interesting property of Eq.(0.1) revealed by the numerical computations is that the above proposition is true at least up to $|\varepsilon| = 1.8$.

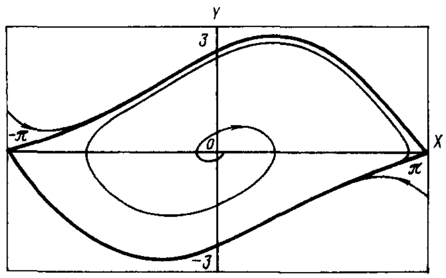


Fig.1

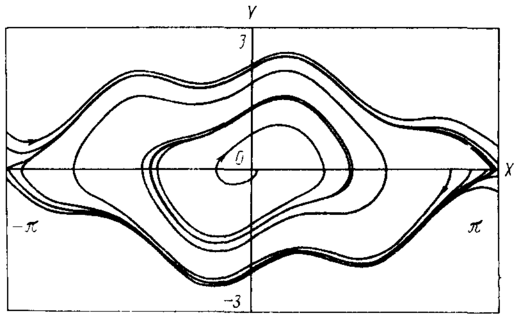


Fig.2

The numerical integration of Eq.(1.5) was accomplished using formulae of the Runge-Kutta family, with permitted error per step $O(H^6)$, where H is the step-size, and requiring only six calls per step on the routine for computing the right-hand side (in the normally used Runge-Kutta formulae the error per step is $O(H^5)$ and the number of calls is 11). The computations were done with both single precision (4 bytes per real number) and double precision. To increase n it was necessary to increase the precision of the computations.

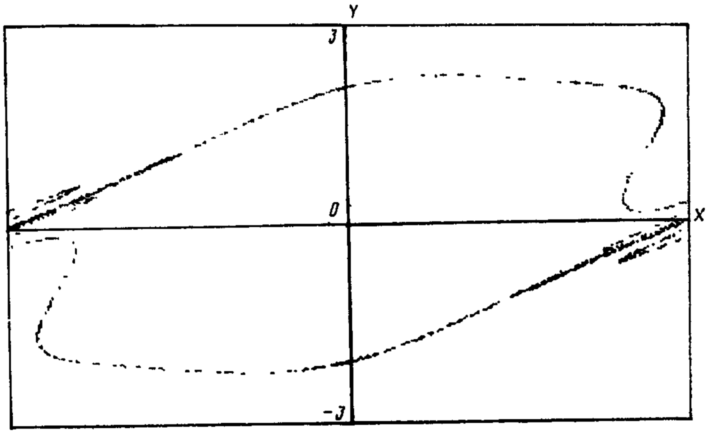


Fig.3

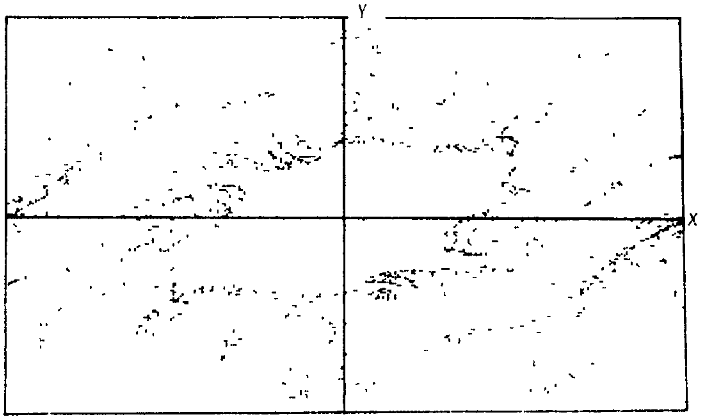
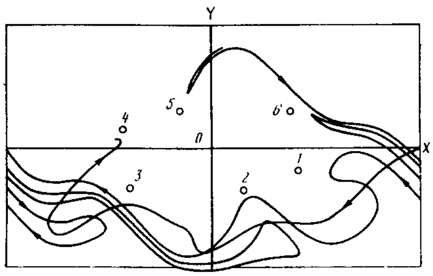


Fig.4



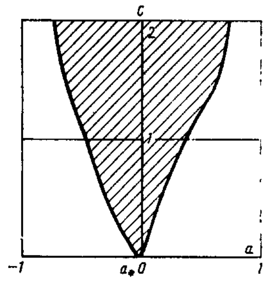


Fig.5 Fig.6

2. Investigation of Eq.(0.1). We will first briefly consider the integrable case, when $\varepsilon = 0$.

The unperturbed equation corresponding to (0.1) has an energy integral

$$x'^2/2 - \cos x = h \tag{2.1}$$

The region of oscillatory motions of the pendulum corresponds to $h \in (-1, 1)$, with $h = -1$ defining a centre-type equilibrium state, and $h = 1$ a saddle. The region of rotary motions of the pendulum corresponds to $h > 1$ Using (2.1), we find

$$h \in (-1, 1), \quad x(\theta, h) = 2 \arcsin(k \sin(\theta/\omega)) \tag{2.2}$$
$$\dot{x} = y(\theta, h) = 2k\, \mathrm{cn}(\theta/\omega)$$
$$k^2 = (1+h)/2, \quad \omega = \pi/(2\mathbf{K})$$
$$h > 1, \quad x(\theta, h) = 2\,\mathrm{am}(\theta/k\omega) \tag{2.3}$$
$$\dot{x} = y(\theta, h) = \pm 2k^{-1}\, \mathrm{dn}(\theta/k\omega)$$
$$k^2 = 2/(1+h), \quad \omega = \pi/(k\mathbf{K}); \ \theta = \omega t$$

Use is made here of the Jacobi elliptic functions, where $\mathbf{K}$ denotes the complete elliptic integral of the first kind with modulus k, and ω is the frequency of the motion.

It is well-known (see, e.g., /10, 11/) that the fundamental question in the investigation of Eq.(0.1) the determination of its l.c.'s - reduces to determining the zeros of the Poincaré-Pontryagin generating function

$$F_n(h) = \int_0^{2\pi} f_0(x, \dot{x})\, x_\theta'\, d\theta = \int_0^{2\pi} \cos(nx)\, \dot{x} x_\theta'\, d\theta \tag{2.4}$$

($x, \dot{x}, x_\theta'$ are defined in (2.2), (2.3)). The elliptic integral in (2.4) can be reduced to standard form /1, 12/

$$F_n^{(s)}(\rho) = C_n^{(s)} [P_n^{(s)}(\rho)\, \mathbf{K}(\rho) + Q_n^{(s)}(\rho)\, \mathbf{E}(\rho)] \tag{2.5}$$
$$C_n^{(1)} = 16/(2n+1)!!, \quad C_n^{(2)} = 8\rho^{-n-1/2}/(2n+1)!!, \quad \rho = k^2 \in (0, 1)$$

where $P_n^{(s)}$, $Q_n^{(s)}$ are polynomials of degree n, $n \geqslant 1$, $\mathbf{E}$ is the complete elliptic integral of the second kind, and the index $s = 1$ indicates the region of oscillatory motions and $s = 2$ that of rotary motions.

Thus, investigation of Eq.(0.1) leads to the study of two classes of special functions: $\{F_n^{(s)}(\rho)\}, s = 1, 2$. For convenience we shall consider $F_n^{(1)}/16$ instead of $F_n^{(1)}$ and $F_n^{(2)}\rho^{1/2}/8$ insteady of $F_n^{(2)}$, while nevertheless retaining the previous notation $F_n^{(s)}$.

Properties of the functions $F_n^{(1)}(\rho)$ (oscillatory region). 1°. The functions $F_n^{(1)}(\rho)$ are solutions of Gauss's linear hypergeometric equation /1, 13/

$$\rho(1-\rho)(F_n^{(1)})'' + \lambda_n F_n^{(1)}/4 = 0, \ \lambda_n = 4n^2 - 1, \quad n = 0, 1, \ldots \tag{2.6}$$

Hence follows the representation

$$F_n^{(1)}(\rho) = C\rho F(1/2 - n, 1/2 + n, 2; \rho) \tag{2.7}$$

where F is a hypergeometric function, $C = \mathrm{const}$.

2°. It follows from (2.5), (2.7) that

$$F_n^{(1)}(0) = 0, \ F_n^{(1)}(1) = (-1)^{n+1}/\lambda_n, \ (F_n^{(1)}(0))' = 4\pi, \ \lim_{\rho \to 1} (F_n^{(1)})' = (-1)^n \infty$$

3°. From property 2° and (2.7) we deduce $C = 4\pi$. The function $F(1/2 - n, 1/2 + n, 2; \rho)$ can be continued analytically into the complex plane C^* slit the real axis from $z = 1$ to $z = \infty$. Then

$$F_n^{(1)}(z) = 4\pi z F(1/2 - n, 1/2 + n, 2; z), \ z \in C^* \tag{2.8}$$

4°. We have the following recurrence formula /7/:

$$(3 + 2n) F_{n+1}^{(1)}(z) + 4n(2z - 1) F_n^{(1)}(z) + (2n - 3) F_{n-1}^{(1)}(z) = 0 \tag{2.9}$$

It follows at once from (2.4) that

$$F_0^{(1)}(z) = (z - 1)\mathbf{K}(z) + \mathbf{E}(z), \quad F_1^{(1)}(z) = [(1 - z)\mathbf{K} + (2z - 1)\mathbf{E}]/3 \tag{2.10}$$

5°. The functions $F_n^{(1)}(z), n \geqslant 1$, have exactly $n - 1$ zeros in the interval $(0, 1)$ of the real axis. The zeros of $F_n^{(1)}(z)$ and $F_{n+1}^{(1)}(z)$ separate one another. The function $F_0^{(1)}(z)$ has no zeros in $(0, 1)$ $(F_0^{(1)}(z) > 0)$.

The proof of this property follows from properties 2°, 4° (see also /7/).

6°. The zeros of the functions $F_n^{(1)}(z)$ in the interval $(0, 1)$ are simple.

Proof. Formula (2.9) implies the representation

$$F_n^{(1)}(z) = q_{n-2}(z) F_0^{(1)}(z) + p_{n-1}(z) F_1^{(1)}(z), \quad n \geqslant 2 \tag{2.11}$$

$$p_{j+1}=\frac{4j+4}{2j+5}(1-2z)\,p_j+\frac{1-2j}{2j+5}\,p_{j-1},\quad j=1,2,\ldots,n-2 \tag{2.12}$$
$$p_0=1,\quad p_1=4\,(1-2z)/5$$
$$q_j=\frac{4j+4}{2j+5}(1-2z)\,q_{j-1}+\frac{1-2j}{2j+5}\,q_{j-2},\quad j=2,\ldots,n-2 \tag{2.13}$$
$$q_0=1/5,\quad q_1=8\,(1-2z)/5$$

Using formulae (2.10), (2.11), we find (putting $z=\rho\in(0,1)$)

$$(F_n^{(1)}(\rho))'=\frac{A_nF_0^{(1)}+B_nF_1^{(1)}}{4\rho(1-\rho)} \tag{2.14}$$
$$A_n=4\rho(1-\rho)\,q_{n-2}'+p_{n-1}+(1-2\rho)\,q_{n-2}$$
$$B_n=4\rho(1-\rho)\,p_{n-1}'+3\,(1-2\rho)\,p_{n-1}+3q_{n-2}$$

Suppose that at some $\rho=\rho_*$ we have $(F^{(1)}{}_n(\rho_*))'=0$. Then by (2.14)

$$F_1^{(1)}(\rho_*)=-(A_n(\rho_*)/B_n(\rho_*))\,F_0^{(1)}\rho_* \tag{2.15}$$

Substituting (2.15) into (2.11), we get

$$F_n^{(1)}(\rho_*)=F_0^{(1)}(\rho_*)\,D_n(\rho_*)/B_n(\rho_*) \tag{2.16}$$
$$D_n=q_{n-2}B_n-p_{n-1}A_n$$

We assert that

$$D_n=-3/\lambda_n \tag{2.17}$$

Indeed, consider D_j, $1<j\leqslant n$. Using the recurrence relations (2.12), (2.13), we obtain $D_2=-1/5=-3/\lambda_2$, $D_3=-3/35=-3/\lambda_3$. Proceeding by induction, we set $D_j=-3\lambda^{-1}{}_j$, $1<j\leqslant n$ and prove that

$$D_{n+1}=-3\lambda_{n+1}^{-1} \tag{2.18}$$

Using (2.6), (2.11), one can set up a system of differential equations for p_{n-1}, q_{n-2}. Differentiation of D_n then gives $D_n'(\rho)\equiv 0$. Hence, by (2.12), (2.13), we obtain (2.18).

Finally, it follows from the condition $D_n\neq 0$ that $B_n(\rho_*)\neq 0$. We then obtain from (2.16), remembering that $F^{(1)}{}_0(\rho_*)>0$ that $F^{(1)}{}_n(\rho_*)$ does not vanish and is finite. This proves property 6°.

7°. By the theorem on alternating zeros /14/, the zeros of $F_n^{(1)}(\rho)$ and $(F_n^{(1)}(\rho))'$ separate one another.

Properties of the functions $F_n^{(2)}(\rho)$ *(rotary region)*. 1°. By the definition of $F_n^{(2)}(\rho)$:

$$F_n^{(2)}(\rho)=\frac{1}{2}\int_\alpha^\beta\cos nx\,\mathrm{dn}^2\varphi\,d\varphi,\quad x=2\,\mathrm{am}\,\varphi \tag{2.19}$$
$$(F_n^{(2)}(\rho))'=-\frac{1}{4}\int_\alpha^\beta\cos nx\,\mathrm{sn}^2\varphi\,d\varphi$$

Throughout, $\alpha=0$, $\beta=2\mathbf{K}$.

2°. $F_n^{(2)}(\rho)$ satisfies the equation

$$\rho\,(1-\rho)\,(\rho\,(F_n^{(2)})')'+(\rho-\mu_n)\,F_n^{(2)}/4=0,\ \mu_n=4n^2,\ n=0,1,\ldots \tag{2.20}$$

Proof. It is more convenient to go back to the original function $F_n(\rho)=8F^{(2)}{}_n(\rho)/\rho^{1/2}$. Then, using (2.19), we obtain

$$(F_n)'=-\frac{2}{\rho^{3/2}}\int_\alpha^\beta\cos nx\,d\varphi$$
$$(F_n)''=\frac{3}{\rho^{5/2}}\int_\alpha^\beta\cos nx\,d\varphi-\frac{2}{\rho^{3/2}}\left[\frac{\mathbf{E}-(1-\rho)\,\mathbf{K}}{\rho\,(1-\rho)}-\int_\alpha^\beta(\cos nx)_\varphi'\,\varphi_\rho'\,d\varphi\right],$$
$$\varphi_\rho'=\frac{1}{2\,(1-\rho)\,\rho^{1/2}}\left[\frac{\mathbf{E}(\mathrm{am}\,\varphi,k)-(1-\rho)\,\varphi}{k}-\frac{k\,\mathrm{sn}\,\varphi\,\mathrm{cn}\,\varphi}{\mathrm{dn}\,\varphi}\right],\quad k=\rho^{1/2}$$

After some reduction, we find

$$(F_n)''=\frac{1}{\rho^{5/2}}\left[4\int_\alpha^\beta\cos nx\,d\varphi-\frac{1-4n^2}{1-\rho}\int_\alpha^\beta\cos nx\,\mathrm{dn}^2\varphi\,d\varphi\right]$$

Using the expressions for $F_n, (F_n)'$, we obtain an equation which in the case of $F_n^{(2)}$ reduces to (2.20).

3°. The solution of Eq.(2.20) satisfying the condition $F_n^{(2)}(0) = 0$ may be expressed in terms of the solution of the hypergeometric equation /14/:

$$F_n^{(2)}(\rho) = C_n \rho^n F(-{}^1/_2 + n, {}^1/_2 + n, 1 + 2n; \rho) \tag{2 21}$$

$$C_n = \pi \frac{(-1)^{n+1}(n + {}^1/_2)[(2n-1)!!]^2}{(\mu_n - 1)2^n (2n)!}, \quad \mu_n - 1 = \lambda_n$$

(the constant C_n is determined by the condition $F_n^{(2)}(1) = F_n^{(1)}(1) = (-1)^{n+1}\lambda_n^{-1}$).

4°. It follows from (2.21) that $F_n^{(2)}$ can be continued analytically into the complex plane C^*:

$$F_n^{(2)}(z) = C_n z^n F(-{}^1/_2 + n, {}^1/_2 + n, 1 + 2n; z), \quad z \in C^* \tag{2 22}$$

5°. It follows from a theorem of Runckel /15/ on the zeros of hypergeometric functions that $F_n^{(2)}(z)$ has no zeros in C^* other than $z = 0$. Hence $F_n^{(2)}(\rho)$ has no zeros for $\rho \in (0, 1)$

6°. Using the recurrent formulae for contiguous hypergeometric functions, we obtain the following recurrent formula:

$$(2n + 3)\, zF_{n+1}^{(2)}(z) + 4n(2 - z)\, F_n^{(2)}(z) + (2n - 3)\, zF_{n-1}^{(2)}(z) = 0, \; z \in C^* \tag{2.23}$$

and moreover, by the first formula (2.19),

$$F_0^{(2)}(z) = \mathbf{E}(z), \; F_1^{(2)}(z) = [2(z-1)\mathbf{K} + (2 - z)\mathbf{E}]/(3z)$$

Property 5° of the functions $F_n^{(s)}(z)$, $s = 1, 2$, property 6° of $F_n^{(1)}(z)$ and a theorem of Pontryagin/10, 11/ imply the truth of the proposition stated in Sect.1.

Remark. The functions $F^{(\,)}{}_n(\rho)$, $\rho \in [0, 1]$, are the eigenfunctions of boundary-value problems for Eqs.(2.6) with $s = 1$ and (2.20) with $s = 2$ The recurrent formulae (2.9), (2.23) furnish an effective way of evaluating these functions by computer.

3. *Investigation of Eq.(1.4).* The generating function for Eq.(1.4) is

$$\Phi_n^{(s)}(\rho; a) = F_n^{(s)}(\rho) + aF_0^{(s)}(\rho), \; n \geqslant 1, \; \rho \in (0, 1) \tag{3.1}$$

Thanks to the obvious property $F_n^{(1)}(1) = F_n^{(2)}(1)$ we can define a global generating function

$$\Phi_n(h; a) = \begin{cases} \Phi_n^{(1)}(h; a), & h \in (-1, 1] \\ \Phi_n^{(2)}(h, a), & h \geqslant 1 \end{cases}$$

This fact, together with formulae (2.7), (2.21), justify our consideration of the functions $F_n^{(s)}(\rho)$ as generating functions in the interval $[0, 1]$.

The parameter a in (3.1) is chosen so that $\Phi_n^{(1)}(1; a) = 0$ Then $a = a_*(n) = (-1)^n\lambda_n^{-1}$

Simple zeros of $\Phi_n^{(1)}(\rho; a_*)$ correspond to transversal points of intersection of the curves $F_n^{(1)}(\rho)$ and $-a_* F_0^{(1)}(\rho)$. Using (2.16), (2.17) and the inequality $|B_n(\rho_*)| < 3$, it is not hard to prove that $|F_n^{(1)}(\rho_*)| > |-a_* F_0^{(1)}(\rho_*)|$, where ρ_* is an extremum point of $F_n^{(1)}(\rho)$.

When this inequality holds, it follows from properties 2°, 5°, 7° and Theorem 1 of /7/ that the function $\Phi_n^{(1)}(\rho; a_*)$ has $n - 1$ simple zeros in $(0, 1)$. In addition, by the choice of a, the function has one more simple zero at $\rho = 1$, which "migrates" into the region $\rho < 1$ $(h < 1)$ as $|a|$ rises above $|a_*|$. If $|a|$ falls below $|a_*|$, then $\Phi_n^{(1)}(\rho, a)$ has $n - 1$ zeros, and $\Phi_n^{(2)}(\rho, a)$ has one zero $\rho = \rho_0$; moreover, $\rho_0 \to 0$ as $|a| \to 0$ (the l.c. in the region of rotary motion goes to infinity).

This theoretical result, valid for small $|\varepsilon|$, is illustrated by our computer results for $\varepsilon = 1, a = a_*$ in Fig.1 $(n = 1)$ and Fig.2 $(n = 3)$.

4. *Investigation of Eq.(1.5).* Unlike the autonomous Eq.(1.4), Eq.(1.5), which is not autonomous, may have resonant periodic regimes with various periods; moreover, the separatrices of manifolds of periodic saddle-type motion may intersect transversally, and as a result a homoclinic structure may exist (a non-trivial hyperbolic set). Under certain conditions there may exist a non-trivial attracting set - a quasi-attractor (q.a.).

An equation of type (1.5) was investigated in /3/ for small $|\varepsilon|$ (see also /2/). Here, therefore, we shall dwell only on the case of relative large $|\varepsilon|$. In that case the computer is the main tool of the investigation.

As we know, the study of non-autonomous time-periodic equations of type (1.5) leads to an investigation of the Poincaré map of the section $t = 0$ into itself over the period of the applied force. At $c = 0$ the invariant curves of this map coincide with the appropriate trajectories of Eq.(1.4) and lie on the cylinder $\{x \bmod 2\pi, y\}$.

We will present the results of a numerical experiment carried out for Eq.(1.5) with $c \neq 0$ and $\nu = 5$.

At $a = a_*(n)$ Eq.(1.4) has a saddle-type l.c., but for some values of ε, c Eq.(1.5) has a q.a. Fig.3 illustrates the q.a. obtained for $n = 1$, $\varepsilon = 1$, $c = 1$, $a = a_* = -1/3$, and Fig.4 a q.a. with $n = 3$, $\varepsilon = 1$, $c = 1$, $a = a_* = -1/35$. Each figure represents some 3,000 points of the Poincaré map. The attracting region of these q.a.'s is the entire phase cylinder. That the trajectories are unstable at a q.a. was determined by computing the Lyapunov characteristic exponents $\mu^\pm$ and fractional part of the Lyapunov dimension $\gamma = |\mu^+/\mu^-|$. For the case $n = 1$ (Fig.3) the result was $\gamma \approx 0.11$, and for $n = 3$ (Fig.4) $\gamma \approx 0.67$. We next put $\varepsilon = 1$.

The idea of obtaining q.a.'s by time-periodic disturbance of a system with stable saddle-type l.c.'s is not new (see, e.g., /16, 17/). The novelty of our approach here is that the method does not always imply the existence of a q.a. For Eq.(1.5) with $n = 3$, $a = a_*$ and c less than $c_* \approx 0.72$ the q.a. disappears, though in this region the autonomous Eq.(1.4) has a stable saddle-type l.c., while Eq.(1.5) has a homoclinic structure (Fig.5). The explanation for this situation is that Eq.(1.4) with $n = 3$, $a = a_*$ has two stable and one unstable l.c.'s. As c falls below c_*, the attracting region is that corresponding to the inner stable l.c. (at $c = 0.5$ this is a stable resonant mode of period $6\,(2\pi/5)$, represented by points 1-6 in Fig.5). The situation remains the same for c decreasing down to $c_{**} \approx 0.025$.

Thus, for $c \in (c_{**}, c_*)$ Eq.(1.5) with $n = 3$, $a = a_*$ has a single attractor, corresponding to the inner stable l.c. of Eq.(1.4). The structure of the attractor depends on the value of c. Three cases have been observed: 1) resonant (single-frequency); 2) non-resonant (two-frequency - the Poincaré map has a smooth closed invariant curve and accordingly Eq.(1.5) has a two-dimensional torus); 3) q.a. (a geometrically smeared annular cycle; see /16/).

A q.a. generated by a saddle-type l.c. will be called a saddle-type q.a.

The fact that there is no saddle-type q.a. in the interval $c \in (c_{**}, c_*)$ may be explained as follows. At such values of c the unstable l.c. of Eq.(1.4) is absorbed by the neighbourhood of a saddle-type q.a. (i.e., the neighbourhood of a homoclinic contour), and this neighbourhood becomes unstable from within.

As c decreases away from c_{**} $(c > 0)$, a saddle-type q.a. reappears. At $c \in (0, c_{**})$ the unstable l.c. of Eq.(1.4) is outside the neighbourhood of the saddle-type q.a.

Note that the numbers $c = c_*$, $c = c_{**}$ are as it were "crisis" bifurcation values of the parameter c.

Fig.6 shows the bifurcation curve in the plane of the parameters (a, c), corresponding to contact of a stable and unstable separatrix of a fixed saddle-point $(\pi, 0)$ for $n = 3$ and $\varepsilon = 1$. The hatched region is the region of existence of a homoclinic structure (the region of transversal intersection of separatrices).

The situation becomes more complicated when n increases. Only knowledge of the l.c.'s of Eq.(1.4) will make it possible to understand the bifurcations due to the appearance and disappearance of q.a.'s in Eq.(1.5). It is in this respect that the proposition of Sect.1 plays a crucial role. Thus, for odd n the q.a.'s bifurcate in accordance with the scenario described above for the case $n = 3$. Naturally, as n increases the number of bifurcations of q.a.'s increases, concurrently with the increase in the number of l.c.'s of Eq.(1.4). In the case of even n there is no saddle-type q.a. at $a = a_*$.

In conclusion, attention should be called to the role of global bifurcations, which lead to the appearance of a saddle-type q.a. with developed chaos at moderate amplitudes of the "applied force" (Fig.4). The bifurcations underlying the classical scenarios of transition to chaos for these equations (for example, the Feigenbaum scenario) are local in nature and require quite high amplitudes of the "applied force" in order to obtain developed chaos.

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A SUPPLEMENT TO LAWDEN'S THEORY*

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An extension of the mathematical model of the motion of a particle of variable mass in a central gravitational field is proposed, based on a discrete flow of the reactive mass and jump-type variation of the direction of the reactive force. The problem of programming the optimal orbital transitions is studied, in the case when, as distinct from /1, 2/, the transit time is fixed. As a result, the possible pieces of optimal transitions, corresponding to impulsive, zero, and intermediate thrust, are described. It is shown that intermediate thrust generates motion along spirals which are not the same as Lawden's spiral.

1. Generalization of the equations of motion of a particle of variable mass in a central gravitational field. We know that the analogue of Meshcherskii's equation in the case of the plane motion of a particle of variable mass in a central gravitational field is

$$r^{\cdot\cdot} = f(r, \chi) + m^{-1}P\cos\theta, \quad f = -\nu r^{-2} + \chi^2 r^{-3} \tag{1.1}$$

$$\psi^{\cdot} = r^{-2}\chi, \ \chi^{\cdot} = rm^{-1}P\sin\theta, \ m^{\cdot} = -c^{-1}P$$

Here, r, ψ are the particle polar coordinates, χ is the sectoral velocity, ν is the gravitational constant, m is the mass of the particle, c is the specific impulse of the thrust P, and the angle θ characterizes the direction of the reactive force (Fig.1).

In the classical sense the operations of differentiation of Eqs.(1.1) are only meaningful for ordinary /3/ (e.g., piecewise continuous) programs $P(\cdot)$, $\theta(\cdot)$ However, some problems

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